Math of Machine Learning Olympiad 2022

May 2022

Solve any six of the following problems to get maximal grade. Up to six best solutions will be graded.

Problem 1

Let $\{x_n\}_{n\geqslant 1}$ be a non-decreasing sequence, such that $0 < x_n < A$ for any $n \in \mathbb{N}$ and some $A > 0$. Investigate the convergence of the series

$$\sum_{n=1}^{\infty} \left(1 - \frac{x_n}{x_{n+1}}\right).$$

Problem 2

Find all such matrices $X, Y \in \mathbb{R}^{d\times d}$, $\det(X) \neq 0$, that satisfy the equality

$$XY - YX = X,$$

or prove that such matrices do not exist.

Problem 3

Show that, for any positive distinct real numbers x, y, z, the following matrix is non-degenerate

$$\begin{pmatrix} 1 & x^{2022} & \log(x) \\ 1 & y^{2022} & \log(y) \\ 1 & z^{2022} & \log(z) \end{pmatrix}.$$

Problem 4

Solve the boundary problem:

$$\begin{cases} x^2 y'' = 6y, \\ y(1) = 1, \\ y(2022) = 2022^3. \end{cases}$$

Problem 5

Let $X_1, \ldots, X_n$ be independent variables with uniform distribution on $[0, \theta]$, $\theta > 0$. Let $X_{(1)} \leqslant \ldots \leqslant X_{(n)}$ be the corresponding order statistics. Find such α that $\alpha X_{(n-1)}$ is an unbiased estimate of θ. The multiplier α may depend on n but must not depend on θ.

Problem 6

Let ξ be an exponential random variable Exp(1). A random variable η has the cumulative distribution function

$$F(y) = \mathbb{E}\left[\xi \mathbb{1}(\xi < e^y)\right].$$

Find the density of $\zeta = e^{4\eta}$.

Problem 7

Let A be a symmetric matrix with positive trace. Prove that there exists such $x \in \{-1, 1\}^d$, that $\langle Ax, x \rangle > 0$.

Problem 8

Let $\xi = (\xi_1, \xi_2)^\top \in \mathbb{R}^2$ be a Gaussian random vector with zero mean and covariance matrix

$$\Sigma = \begin{pmatrix} 4 & 1 \\ 1 & 4 \end{pmatrix}.$$

Find $\mathbb{E}[\xi_1^2 \xi_2^2]$.

Solutions

Problem 1

Answer: the series converges.

Let us consider the partial sums:

$$S_N = \sum_{n=1}^{N} \frac{x_{n+1} - x_n}{x_{n+1}}.$$

The inequalities $x_{n+1} \geqslant x_1 > 0$ yield

$$S_N \leqslant \sum_{n=1}^{N} \frac{x_{n+1} - x_n}{x_1} = \frac{x_N - x_1}{x_1}.$$

Since $\{x_n\}_{n\geqslant 1}$ is a bounded monotonously increasing sequence, it converges. Hence, the sequence of the partial sums $\{S_N\}_{N\geqslant 1}$ converges as well.

Problem 2

Answer: there are no such matrices.

Since X is invertible, the equation is equivalent to the following:

$$XYX^{-1} - Y = I.$$

Observe that, since $\mathrm{Tr}(XYX^{-1}) = \mathrm{Tr}(YX^{-1}X) = \mathrm{Tr}(Y)$, the left-hand side has zero trace whereas the trace of the right-hand is equal to d. Thus, there are no matrices, satisfying this equality.

Problem 3

Without loss of generality, we may sort our values $x < y < z$. Then we proceed with row reductions that does not change the determinant

$$\begin{vmatrix} 1 & x^{2022} & \log(x) \\ 0 & y^{2022} - x^{2022} & \log(y) - \log(x) \\ 0 & z^{2022} - y^{2022} & \log(z) - \log(y) \end{vmatrix} = \begin{vmatrix} y^{2022} - x^{2022} & \log(y) - \log(x) \\ z^{2022} - y^{2022} & \log(z) - \log(y) \end{vmatrix}.$$

By the mean value theorem there are $\theta_1 \in (x, y), \theta_2 \in (y, z)$ such that $\log(y) - \log(x) = \theta_1^{-1}(y - x)$ and $\log(z) - \log(y) = \theta_2^{-1}(z - y)$. Thus,

$$\begin{vmatrix} y^{2022} - x^{2022} & \log(y) - \log(x) \\ z^{2022} - y^{2022} & \log(z) - \log(y) \end{vmatrix} = (y - x)(z - y) \begin{vmatrix} (y^{2022} - x^{2022})/(y - x) & \theta_1^{-1} \\ (z^{2022} - y^{2022})/(z - y) & \theta_2^{-1} \end{vmatrix}.$$

The polynomial $P(x, y) = (y^{2022} - x^{2022})/(y - x) = y^{2021} + y^{2020}x + \ldots + x^{2021}$ takes positive values for all $y > x > 0$. Besides, for any $y > 0$, $P(x, y)$ is strictly increasing in x on $\mathbb{R}_+$. Hence, $P(x, y) < P(z, y)$, whereas $\theta_1^{-1} > \theta_2^{-1}$. This observation yields that

$$\begin{vmatrix} P(x, y) & \theta_1^{-1} \\ P(z, y) & \theta_2^{-1} \end{vmatrix}$$

is non-zero.

Problem 4

Answer: $y = x^3$.

Making the substitution $x = e^t$, $t \in \mathbb{R}$, we obtain that $y' = e^{-t}\dot{y}$, $y'' = e^{-2t}(\ddot{y} - \dot{y})$, and, hence,

$$\ddot{y} - \dot{y} = 6y.$$

The general solution of this ordinary differential equation is given by

$$y = C_1 e^{3t} + C_2 e^{-2t} = C_1 x^3 + \frac{C_2}{x^2}, \quad x > 0.$$

where the constants C_1 and C_2 should be determined from the boundary conditions. The conditions $y(1) = 1$, $y(2022) = 2022^3$ yield

$$\begin{cases} C_1 + C_2 = 1, \\ 2022^3 C_1 + \frac{C_2}{2022^2} = 2022^3. \end{cases}$$

Since the matrix of the linear system is non-degenerate, we obtain that $C_1 = 1, C_2 = 0$ is the unique solution. Hence, $y = x^3$.

Problem 5

Answer: $\alpha = (n+1)/(n-1)$.

The random variable $X_{(n-1)}/\theta$ has the beta-distribution $\mathrm{Beta}(n-1, 2)$. Hence, its expectation is equal to

$$\frac{\mathbb{E}X_{(n-1)}}{\theta} = \frac{n-1}{n+1}.$$

This yields $\alpha = (n+1)/(n-1)$.

Problem 6

Answer: $e^{-t^{1/4}}/(4\sqrt{t})$, $t > 0$.

First, find the cumulative distribution function of the positive random variable ζ. For any $t > 0$, it holds that

$$\mathbb{P}(\zeta < t) = \mathbb{P}\left(\eta < \frac{\ln t}{4}\right) = F\left(\frac{\ln t}{4}\right) = \int\limits_0^{t^{1/4}} x e^{-x} \mathrm{d}x.$$

Differentiating the expression in the right-hand side, we obtain the density of ζ:

$$p_\zeta(t) = \left(x e^{-x}\right)\Big|_{x = t^{1/4}} \cdot \frac{1}{4t^{3/4}} = \frac{e^{-t^{1/4}}}{4\sqrt{t}}, \quad t > 0.$$

Problem 7

Let $\xi_1, \ldots, \xi_d$ be independent uniformly distributed on $\{-1, 1\}$ random variables. Consider the random vector $\xi = (\xi_1, \ldots, \xi_d)^\top$. It holds that $\mathbb{E}\xi = 0$ and $\mathbb{E}\xi\xi^\top = I$. This implies that

$$\mathbb{E}\langle \xi, A\xi \rangle = \mathbb{E}\mathrm{Tr}\left(A\xi\xi^\top\right) = \mathrm{Tr}\left(A\,\mathbb{E}\xi\xi^\top\right) = \mathrm{Tr}(A) > 0.$$

Hence, there exists a realization x of ξ such that $\langle x, Ax \rangle > 0$.

Problem 8

Answer: 18.

Let us make a substitution $\eta_1 = \xi_1 - \xi_2$, $\eta_2 = \xi_1 + \xi_2$. Then the vector $\eta = (\eta_1, \eta_2)^\top$ also has a Gaussian distribution as a linear transformation of a Gaussian random vector. More precisely, it is a centered Gaussian vector with the covariance matrix

$$\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 4 & 1 \\ 1 & 4 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} 6 & 0 \\ 0 & 10 \end{pmatrix}.$$

Since the covariance matrix is diagonal, the components of the Gaussian random vector η are independent. Substituting ξ_1 and ξ_2 by $(\eta_1 + \eta_2)/2$ and $(\eta_2 - \eta_1)/2$, respectively, we obtain that

$$\mathbb{E}\left[\xi_1^2 \xi_2^2\right] = \frac{1}{16}\mathbb{E}\left[(\eta_1 + \eta_2)^2 (\eta_1 - \eta_2)^2\right] = \frac{1}{16}\mathbb{E}\left[(\eta_1^2 - \eta_2^2)^2\right] = \frac{1}{16}\mathbb{E}\left[\eta_1^4 - 2\eta_1^2\eta_2^2 + \eta_2^4\right].$$

Using the equalities $\mathbb{E}\eta_1^4 = 3\left(\mathbb{E}\eta_1^2\right)^2 = 3 \cdot 6^2$, $\mathbb{E}\eta_1^2\eta_2^2 = (\mathbb{E}\eta_1^2)(\mathbb{E}\eta_2^2) = 60$, and $\mathbb{E}\eta_2^4 = 3\left(\mathbb{E}\eta_1^2\right)^2 = 3 \cdot 10^2$, we get that

$$\mathbb{E}\left[\xi_1^2 \xi_2^2\right] = \frac{1}{16}\left(3 \cdot 6^2 - 2 \cdot 60 + 3 \cdot 10^2\right) = \frac{12}{16}\left(9 - 10 + 25\right) = 18.$$