

Calculus

1. Find derivative with respect to x of the function $f(x) = -\ln(\sqrt{e^{2x} + 1}) + e^{-x} \operatorname{arctg}(e^x) + x$

- $e^{-x} \operatorname{arctg}(e^x)$
- $-e^x \operatorname{arctg}(e^{-x})$
- $e^x \operatorname{arctg}(e^x)$
- $-e^{-x} \operatorname{arctg}(e^{-x})$
- $-e^{-x} \operatorname{arctg}(e^x)$

2. Find the definite integral $\int_0^{\pi/2} e^{2x} \cos(x) dx$

- $\frac{e^\pi + 2}{5}$
- $\frac{e^\pi - 2}{5}$
- $\frac{e^{-\pi} - 2}{5}$
- $\frac{e^\pi - 3}{4}$
- $\frac{e^{2\pi} - 1}{5}$

3. Find the following limit: $\lim_{x \rightarrow 0} \frac{e^{\sin 5x} - \cosh \sqrt{10x}}{\ln(1 + 2x^2)}$

- 25/3
- 25/2
- 2/25
- 25/6
- 1/10

4. Calculate the sum of partial derivatives $\partial_x f + \partial_y f + \partial_z f$ of the function $f(x, y, z) = \ln(x^3 + y^3 + z^3 - 2xyz)$ at $x = 1, y = 1, z = 1$

- 3
- 1
- 3
- 2
- 0

5. Find the first five terms (up to x^4) of the Taylor series expansion around $x = 0$ of the function $f(x) = x / \arctan x$

- $1 + \frac{x^2}{3} + \frac{4x^4}{45} + O(x^6)$
- $1 + \frac{x^2}{3} - \frac{8x^4}{45} + O(x^6)$
- $1 + \frac{x^2}{3} - \frac{4x^4}{45} + O(x^6)$
- $1 + \frac{2x^2}{3} + \frac{8x^4}{15} + O(x^6)$
- $1 + \frac{x^2}{3} - \frac{8x^4}{15} + O(x^6)$

6. (2 points) Evaluate the integral

$$\int_{\mathcal{C}} \frac{ydx - xdy}{x^2 + y^2}$$

along the unit circle \mathcal{C} , centered at $x = 2$, $y = 0$.

7. (3 points) All derivatives of the function

$$f(x) = \frac{1}{2} \frac{1+x^2}{1-x^2}$$

at $x = 0$ are integers. How many zeros does $f^{(42)}(0)$ have at the end?

8. (3 points) Consider the largest (in volume) right circular cone fit inside a sphere. What fraction of the volume of the sphere is occupied by this cone?

- 0.2
- 0.25
- 0.3
- 0.35
- 0.4
- 0.45

9. (4 points) The function $x(t)$ solves the differential equation $x'' + xx' = 0$. Determine the initial conditions for which $x(t)$ diverges at some finite $t > 0$.

- $x(0) = 1, x'(0) = -1$
- $x(0) = 5, x'(0) = 0$
- $x(0) = 2, x'(0) = -2$
- $x(0) = 0, x'(0) = 2$
- $x(0) = -2, x'(0) = 2$

Differential equations

10. Determine type of the differential equation $y'' + xy' + x^2y = x^3$

- partial differential equation
- first order differential equation
- linear differential equation with constant coefficients
- linear nonhomogeneous differential equation
- nonlinear homogeneous differential equation

11. Write general solution of the differential equation $y'' + a_1y' + a_0y = 0$

- $Ax^2 + Bx + C$
- $Ax^{\lambda_1} + Bx^{\lambda_2} + Cx^{\lambda_3}$
- $Ax^{\lambda_1} + Bx^{\lambda_2}$
- $Ae^{\lambda_1x} + Be^{\lambda_2x}$
- $Ae^{\lambda x}$
- explicit algebraic form does not exist

12. Solve the differential equation $2xy' + y = 6x$

- $-x + C\sqrt{x}$
- $-x + C/\sqrt{x}$
- $x + C\sqrt{x}$
- $x + C/\sqrt{x}$
- $2x + C\sqrt{x}$
- $2x + C/\sqrt{x}$

13. For the equation $2xy' + y = 6x$ determine $y(4)$ if $y(1) = -2$

- 6
- 3
- 0
- 3
- 6

Linear algebra

14. Let $\mathbf{A} = \begin{pmatrix} 7 & -6 \\ 8 & -7 \end{pmatrix}$. Find \mathbf{A}^{-1} .

$\begin{pmatrix} 7 & 6 \\ -8 & -7 \end{pmatrix}$

$\begin{pmatrix} -7 & -6 \\ 8 & 7 \end{pmatrix}$

$\begin{pmatrix} 7 & -6 \\ 8 & -7 \end{pmatrix}$

$\begin{pmatrix} -7 & 6 \\ -8 & 7 \end{pmatrix}$

None of the above.

15. Let \mathbf{A} , \mathbf{B} , and \mathbf{C} be square invertible matrices of the same size. Simplify the expression

$$(\mathbf{C}^{-2}\mathbf{B}^2)^{-1}\mathbf{C}^{-1}(\mathbf{A}^{-1}\mathbf{B}^{-3}\mathbf{C})^{-1}$$

\mathbf{AB}

\mathbf{BA}

\mathbf{AC}

\mathbf{CA}

\mathbf{CB}

\mathbf{BC}

None of the above.

16. Let \mathbf{A} be a 3×3 real matrix. Characterize its determinant given

$$\det\left(\frac{1}{3} \mathbf{A}^{-1}(\mathbf{A}^T) \mathbf{A}^3\right) = 1,$$

where \mathbf{A}^T is the transpose matrix.

- 0
- 1
- 2
- 3
- 3
- None of the above.

17. Let $\mathbf{A} = \begin{pmatrix} 3 & 0 & 0 \\ -1 & 2 & 0 \\ -2 & 2 & 1 \end{pmatrix}$. Find the largest eigenvalue of \mathbf{A}^{2021} .

- 0
- 1
- 2^{-2021}
- 2^{2021}
- 3^{-2021}
- 3^{2021}
- None of the above.

18. Let \mathbf{A} be a 3×3 real nonsingular matrix such that $\mathbf{A}^{-1} = \mathbf{A}$. Characterize its eigenvalues.

- 2, 1, or -1
- 1 or 2
- 1 or -2
- 1 or -1
- 1, 1, or 2
- None of the above.

Probability

19. If X is a random number in the interval $[0, 2]$ and Y a random number in the interval $[0, 4]$, what is the probability that $X^2 < Y$? Express the result as an irreducible fraction a/b with integer a, b .

20. Which of the following conditions is NOT required for an application of the Central Limit Theorem to a sequence of random variables $(X_n)_{n=1}^N$:

- X_n are independent
- X_n are normal
- X_n are identically distributed
- $N \rightarrow \infty$
- All these conditions are required

21. Suppose that the expectation of a random variable X is 2, and the variance of $2X$ is 3. Find $E(X^2)$. Express the result as an irreducible fraction a/b with integer a, b .

22. Let p_X be the probability density function (p.d.f.) of the random variable X , and p_{X^5} be the p.d.f. of X^5 . If $p_X(1) = 2$, what is $p_{X^5}(1)$? Express the result as an irreducible fraction a/b with integer a, b .

23. Given two random events A and B , suppose that $P(A) = \frac{1}{2}$, $P(A|B) = \frac{1}{5}$, and $P(A \cup B) = 1$. Find $P(B|A)$. Express the result as an irreducible fraction a/b with integer a, b .

Physics

24. (2 points) Two solid dielectric cylinders with the same radius R and permittivities 2 and 5 are placed on large distance in vacuum in a constant electric field directed perpendicular to the cylinders. Find the ratio of induced dipole moments of first and second cylinder.

Answer: $\frac{1}{2}$.

- 1/5
- 1/3
- 1/2
- 1
- 5

25. (3 points) Imagine that Earth is a black body (hopefully it will never happen) and there is no heat generation inside. What would be the average temperature on the Earth due to Sun. Temperature of the Sun surface is 6000 K. The Sun radius is approx $R = 0.7$ million km and Earth is $L = 150$ million km away from the Sun.

- 50 K
- 100 K
- 300 K
- 600 K
- 1000 K

26. (2 points) An air molecule at 25°C and 760 mm pressure travels about 7×10^{-6} cm between successive collisions and moves with a mean speed of about 450 m/s. In the absence of any bodily motion of the air, about how long should it take for a given molecule to move 1 cm from where it is now?

- 10 s
- 3 s
- 17 s
- 21 s
- 0.01 s
- 0.5 s

27. (2 points) The radio transmitter emits 15 W of power at 5200 MHz. How many photons are emitted during one period of electromagnetic wave? Values for some fundamental constants: $h = 6.626 \times 10^{-34}$ J·s, $c = 2.998 \times 10^8$ m/s, $N_A = 6.022 \times 10^{23}$ mol⁻¹

- 1.47×10^5
- 8.37×10^{14}
- 3.41×10^{17}
- 4.12×10^{15}
- 2.26×10^{16}
- 3.49×10^{13}

Chemistry

(1 point) Range the MX compounds adopting the rocksalt-type structure (face-centered cubic cell) according to increasing of their unit cell volume:

1. $\text{SrO} < \text{KBr} < \text{KCl} < \text{MgO} < \text{NaF}$
2. $\text{NaF} < \text{KCl} < \text{KBr} < \text{MgO} < \text{SrO}$
3. $\text{NaF} < \text{MgO} < \text{KCl} < \text{SrO} < \text{KBr}$
4. $\text{MgO} < \text{NaF} < \text{SrO} < \text{KCl} < \text{KBr}$
5. $\text{NaF} < \text{MgO} < \text{KBr} < \text{KCl} < \text{SrO}$

(1 point) What is the correct electron configuration of the Fe^{3+} ion in the octahedral splitting (high-spin)?

1. $t_{2g}^3 e_g^4$
2. $t_{2g}^5 e_g^0$
3. $t_{2g}^2 e_g^3$
4. $t_{2g}^3 e_g^2$
5. $t_{2g}^4 e_g^1$

(2 points) Given LiFePO_4 electrode material particles are of an ideal spherical shape with an average radius of 20 nm, calculate the average thickness of carbon coating if the overall carbon content is 2% mass (consider LiFePO_4 density 3.6 g/cm^3 , carbon density 1.5 g/cm^3).

1. 3 nm
2. 7 nm
3. 10 nm
4. 12 nm
5. 30 nm

(1 point) What volume of 10.0 M H_2SO_4 is required to prepare 4.0 L of 0.50 M H_2SO_4 ?

- 1) 0.20 L
- 2) 0.40 L
- 3) 0.50 L
- 4) 1.0 L

(1 point) Of the choices below, which one is not an ionic compound?

- 1) PCl_5
- 2) FeCl_3
- 3) PbCl_2
- 4) NaCl

(2 points) Calculate the mass of LiPF_6 and ethylene carbonate (EC), and the volume of propylene carbonate (PC) for the preparation of 10 mL of 1M electrolyte (EC:PC 1:1 by vol.), $\rho(\text{EC}) = 1.32 \text{ g/cm}^3$, $\rho(\text{PC}) = 1.20 \text{ g/cm}^3$. Neglect the volume change on adding the salt.

- 1) 1.52 g, 6.66 g, 5 mL
- 2) 15.2 g, 6.66 g, 6 mL
- 3) 1.52 g, 5 g, 5 mL
- 4) 0.152 g, 6.66 g, 5 mL