

Calculus

1. Find derivatives with respect to x of the function $f(x) = (x + 1)^{x+1}$

- $(x + 1)^x$
- $(x + 1)^{x+1}$
- $(1 + \ln(x + 1))(x + 1)^x$
- $(1 + \ln(x + 1))(x + 1)^{x+1}$

2. Find the definite integral $\int_0^{+\infty} x^3 e^{-x^2} dx$

- 1
- 1/2
- 1/3
- 1/4

3. Find partial derivative with respect to y of the function $f(x, y) = e^{\ln(x^2+y^2)}$

- $(2x + 2y)e^{\ln(x^2+y^2)}$
- $2ye^{\ln(x^2+y^2)}$
- $2x + 2y$
- $2y$

4. Find the following limit $\lim_{x \rightarrow 0} \frac{\ln(1 + \sin^2(2x))}{1 - \cos^2(x)}$

- 0
- 1
- 2
- 4
- 8

5. Find the first six terms (up to x^5) of the Taylor series expansion around $x = 0$ of the function $f(x) = \cos \frac{x^2}{2}$
- $1 - \frac{1}{2}x^2 + \frac{1}{4}x^4 + O(x^6)$
 - $1 - \frac{1}{2}x^2 - \frac{1}{4}x^4 + O(x^6)$
 - $1 - \frac{1}{4}x^4 + O(x^6)$
 - $1 - \frac{1}{8}x^4 + O(x^6)$
6. Calculate $\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} dx dy \exp(-5x^2 + 8xy - 5y^2)$. Hint: $\int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}$.
- $\pi/2$
 - $\pi/3$
 - $\pi/4$
 - $\pi/5$
 - $\pi/6$
 - $\pi/8$

Differential equations

7. Determine type of the differential equation $y'' - 2y' + y = \sin x$
- partial differential equation
 - first order differential equation
 - linear differential equation with constant coefficients
 - linear nonhomogeneous differential equation
 - nonlinear homogeneous differential equation
8. Write general solution of the differential equation $x^2y'' + xy' + a^2y = 0$
- $Ax^2 + Bx$
 - $Ax^a + Bx^{-a}$
 - $Ax^{ia} + Bx^{-ia}$
 - $Ae^{ax} + Be^{-ax}$
 - $Ae^{iax} + Be^{-iax}$
 - explicit algebraic form does not exist

9. Solve the differential equation $y' \tan x + y = 4 \sin x$

- $(C + \cos x) \sin x$
- $(C - \sin x) \cos x$
- $(C + \sin 2x) / \cos x$
- $(C - \cos 2x) / \sin x$
- $C \sin x + 1 / \cos x$
- $C \cos x - 1 / \sin x$

10. For the equation $y' \tan x + y = 4 \sin x$ determine $y(\pi/2)$ if $y(0) = 0$

- -2
- -1
- 0
- 1
- 2

Linear algebra

11. Let \mathbf{A} and \mathbf{B} be matrices over \mathbb{R} of size $n \times n$. Choose the correct identities

1. \mathbf{A} is invertible and λ is an eigenvalue of \mathbf{A} , then $1/\lambda$ is an eigenvalue of \mathbf{A}^{-1}
 2. $\det(\lambda\mathbf{A}) = \lambda \det(\mathbf{A})$, $\lambda \in \mathbb{R}, \lambda \neq 0$
 3. $\det(\mathbf{A}^k) = (\det(\mathbf{A}))^k$
 4. Let $\text{rank}(\mathbf{A}) = \text{rank}(\mathbf{B}) = 2$, then $\text{rank}(\mathbf{AB}) = 2$
 5. \mathbf{A} and \mathbf{B} are invertible, then $\mathbf{A} + \mathbf{B}$ is invertible
- 1, 5
 - 2, 4
 - 1, 3
 - 2, 5
 - 2, 3, 5

12. Let $\mathbf{A} = \begin{bmatrix} 3 & 5 \\ 1 & 2 \end{bmatrix}$. Find \mathbf{A}^{-1}

- $\mathbf{A}^{-1} = \begin{bmatrix} -4 & 5 \\ 2 & 3 \end{bmatrix}$
- $\mathbf{A}^{-1} = \begin{bmatrix} 10 & -8 \\ -5 & 6 \end{bmatrix}$
- $\mathbf{A}^{-1} = \begin{bmatrix} 2 & -5 \\ -1 & 3 \end{bmatrix}$
- $\mathbf{A}^{-1} = \begin{bmatrix} 1 & 4 \\ -2 & 3 \end{bmatrix}$
- $\mathbf{A}^{-1} = \begin{bmatrix} 1 & 3 \\ 4 & -6 \end{bmatrix}$

13. Find $\det \left(\begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & 3 & 4 & 0 \\ 0 & 0 & 5 & 6 \\ 1 & 0 & 0 & 7 \end{bmatrix} \right)$

- 14
- 33
- 57
- 132
- 192

14. Let \mathbf{A} , \mathbf{B} and \mathbf{C} be $n \times n$ invertible matrices. Simplify the expression $\mathbf{B}(\mathbf{A}\mathbf{C}^{-1})^{-1}(\mathbf{C}^2\mathbf{A}^{-1})^{-1}\mathbf{C}^2$

- \mathbf{A}
- \mathbf{B}
- \mathbf{BC}
- $\mathbf{C}^{-1}\mathbf{BC}$
- $\mathbf{C}^{-1}\mathbf{A}^{-1}\mathbf{BC}^{-1}\mathbf{A}\mathbf{C}^2$

15. Let $\mathbf{A} = \begin{bmatrix} 2 & -1 & -1 \\ 0 & -1 & 0 \\ 0 & 2 & 1 \end{bmatrix}$. Find the maximal eigenvalue of the matrix \mathbf{A}^7

- 16
- 32
- 64
- 128
- 256

Probability

16. An urn contains two white and two black balls. We draw balls from the urn randomly and stop after we find a black ball. What is the expectation of the total number of the drawn balls?

- $1\frac{1}{2}$
- $1\frac{2}{3}$
- 2
- $2\frac{1}{3}$
- $2\frac{1}{2}$

17. Let a, b, c be three independent random variables uniformly distributed in the segment $[0, 1]$. What is the probability that $a > b + c$?

- $\frac{1}{2}$
- $\frac{1}{3}$
- $\frac{1}{4}$
- $\frac{1}{6}$
- $\frac{1}{12}$

18. Let η and ξ be two independent normal random variables with mean 1 and variance 2. Which of the following statements is correct?

- $\eta + \xi$ and $\eta - \xi$ are uncorrelated and independent
- $\eta + \xi$ and $\eta - \xi$ are uncorrelated, but not independent
- $\eta + \xi$ and $\eta - \xi$ are correlated, but independent
- $\eta + \xi$ and $\eta - \xi$ are correlated and not independent
- None of the statements is correct

19. Patients of a clinic are tested for a particular disease. For each patient, the result of the test – ‘infected’/‘not infected’ – is correct with the probability 0.8. Suppose that 20% of the patients are infected. What is the probability that a given patient is indeed infected if his/her test result shows ‘infected’?

- 0.4
- 0.5
- 0.6
- 0.64
- 0.8

Physics (counted for Materials Science program)

20. (2 points) Estimate the energy splitting between the lowest two levels for an electron in a three-dimensional cube-shaped potential box with the side length 10 nm.

- 1 meV
- 10 meV
- 100 meV
- 0.11 eV
- 0.22 eV
- 1.12 eV

Chemistry (counted for Materials Science program)

21. (4 points) Hydroboration of styrene followed by oxidation with hydrogen peroxide leads to the formation of compound A. Oxidation of styrene with potassium permanganate in an acidic medium results in compound B. Compounds A and B can react with each other under acidic catalysis to form compound C. What is the volume of carbon dioxide (molar volume of ideal gas $V_m = 22.4$ L/mol), released during the complete combustion (in excess of O_2) of 5.00 grams of compound C?

- 7.00 L
- 7.43 L
- 7.67 L
- 8.02 L
- 8.31 L